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Chapter 3 – Vector Spaces

3.1 – Vector Spaces and Fields

Set of rules called Field Axioms:

- All the same properties as vector space
- Additive Inverse: There exists for all a, a vector –a such that a+(-a) = (-a)+a = 0
- Multiplicative Inverse: There exists for all a, a vector a^{-1} such that $aa^{-1} = a^{-1}a = 1$ if a != 0

Properties to define a set of vectors as a vector space:

- Closed under addition ($V_1 + V_2 \in V$)
- Closed under scalar multiplication ($2V_1 \in V$)
- (u+v)+w = u+(v+w) **A1**
- v+w = w+v **A2**
- 0+v = v **A3**
- v+(-v) = 0 **A4**
- r(v+w) = rv+rw **S1**
- (r+s)v = rv+sv **S2**
- r(sv) = (rs)v **S3**
- 1v = v **S4**

Elementary Properties of Vector Spaces

1. Vector 0 is unique vector x satisfying x+v = v for all vector v in V
2. For each vector v in V, the vector –v is the unique vector y satisfying v + y = 0
3. If u+v=u+w, wherein u,v,w ∈ V, then v=w
4. 0v = 0 for all vectors in V
5. r0 = 0 for all scalars in R
6. (-r)v = r(-v) = -r(v) for all scalars r in R and vectors v in V

3.2 – Basic Concepts of Vector Spaces

Linear Combinations:

Given vectors $v_1, v_2, \ldots, v_k$ in a vector space V and scalars $r_1, r_2, \ldots, r_k$ in R, the vector
$r_1v_1+r_2v_2+\ldots+r_nv_n$ is a linear combination of the vectors v… with scalar coefficients r…

Spans:

X is a subset of V, the span of X is the set of all linear combinations of vectors in X, sp(X).
If W = sp(X) then the vectors in X span or generate W
If V = sp(X) for some **finite** subset X of V, then V is finitely generated

Subspaces:

A subspace W of a vector space V is a subspace if W fulfills the requirements of a Vector Space as well.

Independence:

X is a set of vectors in V, if there exists a $r_1v_1 + r_2v_2 + r_kv_k = 0$ wherein r_j != 0, if such a dependence holds, then X is linearly dependent, otherwise it is linearly independent
To find the linear dependency of a matrix, we simply check if the determinant of the matrix represented by the column vectors in V is 0 (0 = LI, otherwise LD)

Bases and Dimension

A base b is a basis for V if:

1. Set of vectors in b spans V, or sp(b) = V
2. Set of vectors is linearly independent

Dimensions of bases for the same V is the same.
Dimensions refer to the number of vectors in the span.
Generating/Extending a basis:
Create a Matrix A with your vectors and the elementary vectors (all in columns) and reduce to row-echelon form and remove LDs.

3.3 – Coordinatization of Vectors

Ordered Bases:

$(e_1,e_2\ldots e_n)$ is the standard ordered basis for R^n
Instead of talking about sets such as $\{b_1, b_2\}$ because that would equal $\{b_2, b_1\}$, we can use an ordered set like (b_1,b_2)

Coordinatization of Vectors:

Every v in V can be represented by $r_1b_1+r_2b_2\ldots$, we call the set of unique scalars $[r_1,r_2\ldots r_n]$ the Coordinatization of v relative to B wherein B is a basis for V and $(b_1, b_2\ldots b_n)$ is an ordered basis.
We can also calculate if things are independent in the vector space P_2 if we take B = $(x^2, x, 1)$ and row-reduce the matrix represent
General Solution: We take B = (decreasing/increasing set of values i.e. $1,x,x^2$ or 1, sin(x), sin(2x)) then form matrix & solve using an augmented matrix with the augmented side as our vector so [100..|vector]

3.4 – Linear Transformations

Linear Transformations must follow the below properties:

- T(u+v) = T(u) + T(v) **[Preservation of addition]**
- T(ru) = rT(u) **[Preservation of scalar multiplication]**

T: V → V' is to say that the linear transformation T maps from the domain V to the codomain V'
If W is a subset of V, then T{W} = {T(w) | w ∈ W} is the image of W under T. T[V] is the **range** of T.
If W' is a subset of V', then T^{-1}\{W'} = {v ∈ V | T(v) ∈ W'} is the inverse image of W' under T. T^{-1}\{0'} is the **kernel** of T. (all v ∈ V maps to 0')
The equation T(x) = b
Ker(T) is the subspace of V is the solution set of the homogeneous transformation equation T(x) = 0.
A Linear Transformation is **One to One:** If ker(T) is zero, then T(x) = b has at most one solution, and so T is one-to-one
T: V→V' is **an invertible transformation** if T^{-1} . T is the identity transformation on V and T . T^{-1} is the identity transformation on V'
Invertible Linear Transformations Must Satisfy:
One to one: If v_1 != v_2 then $T(v_1)$!= $T(v_2)$ That is, **T is one-to-one if ker(T) = 0**
Onto: If v' is in V', then T(v) = v' for some v in V That is, **T is onto if range(T) = dim(T)**
Isomorphism
An isomorphism is a linear transformation T: V→V' that is one-to-one and onto V'.
If isomorphism T exists, then it is invertible and its inverse is also an isomorphism
V and V' are said to be isomorphic vector spaces
Matrix Representation of Transformations
A is the standard matrix where jth column is the column vector of $T(e_j)$ where e is the coordinate vector relative to B for the b_jth ordered basis in B.
Matrix Rep of T^{-1} is the inverse of the matrix rep of T relative to B, B'

3.5 – Inner Product Spaces

The inner product on a vector space V is a function that associates each pair of vectors v, w in V with a real number, written <v, w> satisfying all u, v, w in V for all scalars r:

- <v, w> = <w, v>
- <u, v+w> = <u, v> + <u, w>
- r<v, w> = <rv w> = <v, rw>
- <v, v> >= 0 and <v, v> = 0 iff v = 0

Inner Product Space is a vector space V together with an inner product on V.
Magnitude:
The magnitude or norm of a vector v in a n inner product space V is ||v|| = sqrt(<v, v>)
Also we have that
||rv|| = |r| ||v|| (can remove a scalar)
Schwarz Inequality

Schwarz inequality: $|\langle \mathbf{v}, \mathbf{w}\rangle| \leq \|\mathbf{v}\|\,\|\mathbf{w}\|,$
Triangle inequality: $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$

Chapter 4 – Determinants

4.4 – Linear Transformations and Determinants

We have the volume of any n-box defined as **V = sqrt(det(A^tA))**
For a transformation T, we have **the Rate of Volume Change as det(A)** where A is the standard matrix representation of T
Volume of G in R^n under transformation T is equal to **sqrt(det(A^TA)) * V**

Chapter 6 – Orthogonality

6.1 – Projections

The projection **p** of **b on sp(a) is: p = [(b . a)/(a . a)]a**
The **orthogonal complement of a subspace** is gotten by using the generating set as **ROW** vectors, then finding the null-space of A.
We can use **cross prod** v_1 x v_2 to find a vector orthogonal to both vectors. This is **u x v = (u_2v_3 – u_3v_2)i - (u_3v_1 – u_1v_3)j + (u_1v_2 – u_2v_1)k**
To find the projection of b on a subspace W, we have:

1. Select a basis of {v_1…v_n} (usually given)
2. Find a basis for total of W or W^T usually by cross-product or null-space of W generating set row matrix
3. Set {v_1, v_2, … v_n} as column vectors, then find augmentation of b into an identity matrix. Let this augmentation be called r
4. We can then solve for b_w = r_1v_1 + r_2v_2… r_nv_n

6.2 – The Gram-Schmidt Process

If we know a base is orthogonal, we can simply compute b_w using:
b_w = ((b.v_1/v_1.v_1)v_1) + ((b.v_2/v_2.v_2)v_2) + … + ((b.v_n/v_n.v_n)v_n)
We can create an orthonormal basis by finding an unit vector for each orthogonal basis vector, such as ||v_n|| = 1.
This way, we can set instead of the above equation, **b_w = ((b.v_1)v_1) + ((b.v_2)v_2) + … + ((b.v_n)v_n)** since v_n . v_n will always be 1.
Gram-Schmidt Theorem: Let W be a subspace of R^n {a_1…a_n} being a basis for W. There exists an orthonormal basis
We have the general **Gram-Schmidt Formula as:**
v_j = a_j – ((a_j . v_1 / v_1 . v_1)v_1 + ... + (a_j . v_{j-1}/ v_{i-1} . v_{i-1})v_{j-1}))
Of course, we can **normalize the Gram-Schmidt Formula to become:**
v_j = a_j – ((a_j . v_1)v_1 + ... + (a_j . v_{j-1})v_{j-1})

6.3 – Orthogonal Matrices

A Matrix is orthogonal if **(A^TA) = I**. These conditions follow if:

- Its rows form an ortho**normal** basis for R^n
- Its columns form an ortho**normal** basis for R^n
- The matrix is orthogonal – $A^{-1} = A^T$

For any symmetric matrix n x n A, we can have $D = C^{-1}AC$ wherein D is a diagonalization of the matrix, and C is an orthogonal mult

We **can choose C as our diagonalization matrix** by finding the eigenvalues of A, plugging them back into A, finding the eigenvectors of A (null space) and then putting those together.

We can find the orthogonal diagonalization of A by reducing our C into an orthogonal matrix (read: orthonormal)

6.4 – The Projection Matrix

The projection of **b_w of b on the subspace of W is $b_w = (A(A^TA)^{-1}A^T)\, b$**

We can have a projection matrix as **$P = A(A^TA)^{-1}A^T$.**

We have **P satisfying two properties:**

$P^2 = P$ idempotent

$P^T = P$ symmetric

We also have another special case, when W = {a_1, a_2, …} is an orthonormal basis, we can have **$P = AA^T$**

Chapter 7 – Change of Basis

7.1 – Coordinatization and Change of Basis

If we are to change bases, from B = {b_1, b_2…} to B' = {b_1, b_2}, we can represent B and B' as matrices M_B and $M_{B'}$ so that

$v_{B'} = M_{B'}^{-1}M_Bv_B$

or rewritten **$v_{B'} = Cv_B$** wherein $C = M_{B'}^{-1}M_B$. We write this as $C_{B,B'}$ - the **change of coordinates matrix from B to B'**

To compute the COCM, we place B' in the LHS, and B in the RHS of an augmented matrix. We reduce B' to I and modified B is our COC.

7.2 – Matrix Representation and Similarity

We can set up an augmented matrix to transfer from $R_B = C^{-1}AC$ by having LHS = b_1b_2… as column vectors, and $T(b_1)T(b_2)$ on the RHS

By row-reducing $M_B|M_{T(B)}$ we obtain R_B as our right hand side when LHS is reduced to I

Similiarity of Matrices:

Given that $R = C^{-1}AC$, we have that:

1. Eigenvalues of R are the same as eigenvalues of A
2. Algebraic and geometric multiplicity of each eigenvalue is the same as A for each eigenvalue in R
3. If v is an eigenvector in A, then $C^{-1}v$ is an eigenvalue in R

Chapter 8 – Eigenvalues, Further Applications and Computation

8.1 – Diagonalization of Quadratic Forms

Every quadratic form in n variables can be written as x^TUx, where x is the column vector of variables and U is a nonzero upper matrix

So we can have something like:

$[x, y, z]\begin{bmatrix} 1 & -2 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ which translates to $x^2 - 2xy + 6xz + z^2$ in the form of matrix product

Steps for diagonalization of a quadratic form:

1. Find the symmetric coefficient matrix A
2. Find the eigenvalues of A, then the eigenvectors
3. Find the orthonormal basis C of the eigenvectors
4. If we have det(C) = 1, it is a rotation. Otherwise, change signs of one column in C to have det(C) = 1 if det(C) = -1
5. This substitution transforms x = Ct to the form from f(x) to diagonal

Ultimately, we can then read each x,y,z… as row vectors, so that $x = (t_{1,1} - t_{1,2} + t_{2,3})$ and so on.

Chapter 9 – Complex Scalars

9.1 – Algebra of Complex Numbers

Fundamental Theory of Algebra – Every polynomial with coefficients in C has n solutions in C, wherein n is the degree of the polynomial and solutions are counted with their algebraic multiplicity

(a+bi) +/- (c+di) = (a +/- c) + (b +/- d)i

Modulus of z = a + bi = |z| = sqrt(a^2+b^2)

Complex Conjugate z = a+bi is z^ = a-bi

zz^ = (a+bi)(a-bi) = $a^2+b^2 = |z|^2$

w/z = 1/($|z|^2$)(wz^)

Polar Form of Complex Numbers

z = r(cosO + i sin O)

9.2 – Matrix and Vector Spaces with Complex Scalars

We have u, v ∈ C then we can assume that u – <v,u>/<v,v>v is perpendicular to v

Conjugate Transposes –Let A = [a_{ij}] be a m x n matrix.

Conjugate of (A) = m x n matrix $\underline{A}$ = [$\underline{a_{ij}}$] Wherein we define a conjugate as x – yi ➔ x + yi (switch the sign!)
Conjugate Transpose of (A) = A* = $\underline{A}^T$
We have the following properties of a Conjugate Transpose:
(A*)* = A
(A+B)* = A* + B*
(AB)* = B*A*
(zA)* = $\underline{z}$(A*)
A square matrix U is **Unitary** if U*U = I
A square matrix H is **Hermitian** if H* = H

9.3 – Eigenvalues and Diagonalization

We can prove that for every Hermitian matrix, it is diagonalizable by an unitary matrix
Just like in 6.3 we can choose our C by having the eigenvector span as our column vectors for the matrix C.
We can call **A and B unitarily equivalent if B = C^{-1}AC**
Schur's Lemma → Letting A be an n x n complex matrix, there is an unitary matrix U such that $U^{-1}AU$ is upper-triangular
Normal Matrices → A matrix is normal **if its conjugate transpose commutes with itself, that is, A*A = AA***
A matrix must be normal to be unitarily diagonalizable

9.4 – Jordan Canonical Form

Jordan Block – Any matrix where diagonals are same value, and 1s appear on top of the diagonal
Any m x m Jordon Blocks have the following properties:

1. $(J - I)e_i = e_{i-1}$ and $(J - I)e_1 = 0$
2. $(J - I)^m = 0$ except for any non m powers
3. $Je_i = e_i + e_{i-1}$ whereas $Je_1 = e_1$

The definition of a canonical Jordan canonical form is blocks of Jordan Blocks following each other closely
A Jordan Canonical form can be computed if we know eigenvalues of A and the rank of $(A - I)^k$ for each lambda and all pos k.
Every square matrix M has a Jordan canonical form, that is, it is similar to a Jordan canonical form

Questions and Answers

3.3 – Coordinatization of Vectors

Find the coordinate vectors of [1, −1] and of [−1, −8] relative to the ordered basis $B = ([1, -1], [1, 2])$ of $\mathbb{R}^2$.

We see that $[1, -1]_B = [1, 0]$, because

$$[1, -1] = 1[1, -1] + 0[1, 2].$$

To find $[-1, -8]_B$, we must find r_1 and r_2 such that $[-1, -8] = r_1[1, -1] + r_2[1, 2]$. Equating components of this vector equation, we obtain the linear system

$$\begin{aligned} r_1 + r_2 &= -1 \\ -r_1 + 2r_2 &= -8. \end{aligned}$$

The solution of this system is $r_1 = 2$, $r_2 = -3$, so we have $[-1, 8]_B = [2, -3]$. Figure 3.1 indicates the geometric meaning of these coordinates. ■

6.2 – The Gram-Schmidt Process

EXAMPLE 6 Find an orthonormal basis for the subspace

$$W = \text{sp}([1, 2, 0, 2], [2, 1, 1, 1], [1, 0, 1, 1])$$

of $\mathbb{R}^4$.

SOLUTION First we find an orthogonal basis, using formula (6). We take $\mathbf{v}_1 = [1, 2, 0, 2]$ and compute $\mathbf{v}_2$ by subtracting from $\mathbf{a}_2 = [2, 1, 1, 1]$ its projection on $\mathbf{v}_1$:

$$\mathbf{v}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = [2, 1, 1, 1] - \frac{6}{9}[1, 2, 0, 2] = \left[\frac{4}{3}, -\frac{1}{3}, 1, -\frac{1}{3}\right].$$

To ease computations, we replace $\mathbf{v}_2$ by the parallel vector $3\mathbf{v}_2$, which serves just as well, obtaining $\mathbf{v}_2 = [4, -1, 3, -1]$. Finally, we subtract from $\mathbf{a}_3 = [1, 0, 1, 1]$ its projection on the subspace $\text{sp}(\mathbf{v}_1, \mathbf{v}_2)$, obtaining

$$\begin{aligned}\mathbf{v}_3 &= \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\ &= [1, 0, 1, 1] - \frac{3}{9}[1, 2, 0, 2] - \frac{6}{27}[4, -1, 3, -1] \\ &= \left[-\frac{2}{9}, -\frac{4}{9}, \frac{3}{9}, \frac{5}{9}\right].\end{aligned}$$

Replacing $\mathbf{v}_3$ by $9\mathbf{v}_3$, we see that

$$\{[1, 2, 0, 2], [4, -1, 3, -1], [-2, -4, 3, 5]\}$$

is an orthogonal basis for W. Normalizing each vector to length 1, we obtain

$$\left\{\frac{1}{3}[1, 2, 0, 2], \frac{1}{3\sqrt{3}}[4, -1, 3, -1], \frac{1}{3\sqrt{6}}[-2, -4, 3, 5]\right\}$$

as an orthonormal basis for W. ■