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MATB41 –Calculus of Multiple Variables University of Toronto Scarborough, Fall 2016

Chapter 1 – The Geometry of Euclidean Space

1.1 – Vectors in 2D and 3D Space

Parametric Equations: We have the line **L** and the points **P, Q**
$x = x_p + (x_q - x_p)t$
$y = y_p + (y_q - y_p)t$
$z = z_p + (z_q - z_p)t$
So we have L = (x, y, z)

1.2 – Inner Product, Distance, and Length

Nothing of note – basic MATA23/MATB24 Review

1.3 – Matrices, Determinants, and Cross Products

Nothing of note – basic MATA23/MATB24 Review

1.4 – Cylindrical and Spherical Coordinates

To be determined…

1.5 – N-Dimensional Euclidean Space

Nothing of note – basic MATA23/MATB24 Review

Chapter 2 – Differentiation

2.1 – The Geometry of Real Valued Functions

Level Sets – f(x,y,z) where f is a constant value. Thus, imagine a level set as a cut of a function horizontally where for ex. $x^2+y^2 = 1$
We can draw **level curves** to model the appearance of a function by choosing f(x,y,z) = c for several different c values

2.2 – Limits and Continuity

We have the idea of **Open Sets** – a set where for every point x_0 in U, there exists some r > 0 that $D_r(x_0)$ is contained within U
Boundary Point – A point is a boundary point of set A if every neighborhood of x contains a point IN and NOT IN A
Continuity – We say something is continuous if limit of ($x \rightarrow x_0$) f(x) = $f(x_0)$

2.3 – Differentiation

Partial Derivatives – Taking the derivative in respect to one of the multiple variables. In these notes, referred to as pd_x
The Linear Approximation – We have the linear approximation to a plane z = ax + by + c as approximated (affine approximation):

z = $f(x_0, y_0) + pd_x(x_0, y_0)[x - x_0] + pd_y(x_0, y_0)[y - y_0]$

This defines the equation of the **plane tangent** to the graph at the point (x_0, y_0)

Differentiability: We say that x is differentiable if pd_x and pd_y are defined, and as $(x, y) \rightarrow (x_0, y_0)$, we have f(x, y) – **z** (noted above) = 0
That is, the equation z is a good approximation to the function f
Matrix of Partial Derivatives:
We have an idea of a matrix of partial derivatives, $Df(x_0)$ where for every function 1… n, we have that each:
Row = f_1/x_i where x is each variable. For example, given the function (x, x^2, y)
row 1 = [1, 0] row2 = [2x, 0], row3 = [0, 1]
Column = f_i/x_1 where f_i is each function, so for (x, x^2, y)
col1 = [1, 2x, 0] col2 = [0, 0, 1]
Gradient: We have the gradient of a function f by having the vector of [pd_x, pd_y, pd_z…] for every variable x, y, z… in f.

2.5 – Properties of the Derivative

The Chain Rule → D(f . g) = $Df(y_0)Dg(x_0)$. That is, we take f(g(x)) as the product of the matrix of partial derivatives of f(x) with g(x)
Remember that we have to evaluate the matrices for f = f(g(x,y)) and g = g(x,y) then take their product

2.6 – Gradients and Directional Derivatives

A **directional derivative** is obtained by taking the dot product of Grad(f) and the vector **unit vector v**
The **gradient** also points to the point of fastest increase at a given point (x, y…)
The tangent of a point (x, y, z) is calculated by the dot product of Grad(f) with $(x - x_0, y - y_0, z - z_0)$ as defined above in Chapter 2.3

Chapter 3 – High-Order Derivatives: Maxima/Minima

3.1 – Iterated Partial Derivatives

We can have iterated partial derivatives, such as pd_x^2 by taking the derivative of x with respect to pd_x already, same for y, z, etc.
Equality of Mixed Partials: The partial derivatives $f^2/d_xd_y = f^2/d_yd_x$

3.2 – Taylor's Theorem for Multiple Variables

Fuck this shit tbh

3.3 – Extrema of Real-Valued Functions

Local Minimum → For all x in the neighborhood of x_0, we have $f(x) >= f(x_0)$
Local Maxima → For all x in the neighborhood of x_0, we have $f(x) <= f(x_0)$
A **critical point** is defined if f is **not differentiable** or $Df(x_0) = 0$

Any critical point that is not a local maxima or local minima is a **saddle point**
First Derivative Test → If the first derivative $Df(x_0) = 0$ then x_0 is a critical point
Second Derivative Test → Use FDT above to find critical points. We then construct a hessian matrix, and using the Hessian Matrix, we find the determinant. If $\det(Hf(x_0)) < 0$ then it is a **saddle point.** If $\det(Hf(x_0)) > 0$ then it is a **relative extrema.** Otherwise, inconclusive.
Finding Relative Extrema Nature → We then take a look at pd_x^2, if it is > 0 then it is a **local minimum,** otherwise, it is a **local maximum.**
Hessian Matrix → A matrix constructed by having a function f(x, y, z):

$$\begin{matrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{matrix}$$

We then take the values of the critical points of f(x, y, z) and calculate the values of each derivative above. If $\det(Hf(x_0))$!= 0 then the SDT gives us the nature of the critical point.
If the Hessian Matrix is **positive definite** (all eigenvalues are positive) **then it is a local minima**
Elsewise, if the Hessian Matrix is **negative definite** (

3.4 – Constrained Extrema and LaGrange Multipliers

To use LaGrange multipliers, we have to take the function f(x, y, z) and the constraint g(x, y, z) = c and produce:
F(x, y, z, L) = f(x, y, z) – L(g(x, y, z) – c))
So for example, given that we have $f(x, y) = 6x^2 + 12y^2$ and $g(x, y) = x + y = 90$ we can construct:
F(x, y, L) = $6x^2 + 12y^2 - L(x + y - 90)$
$pd_x = 12x - L$ → $x = L/12$
$pd_y = 24y - L$ → $y = L/24$
$pd_L = x + y - 90$ → $L/12 + L/24 - 90 = 0$ → $-3L/24 = -90$ → $L = 720$
We equate each of these = 0, then solve for Lambda (L) and plug back in L into x, y to find our extrema

Chapter 5 – Double and Triple Integrals

5.1 – An Introduction

The notes just tapered off here… lol.